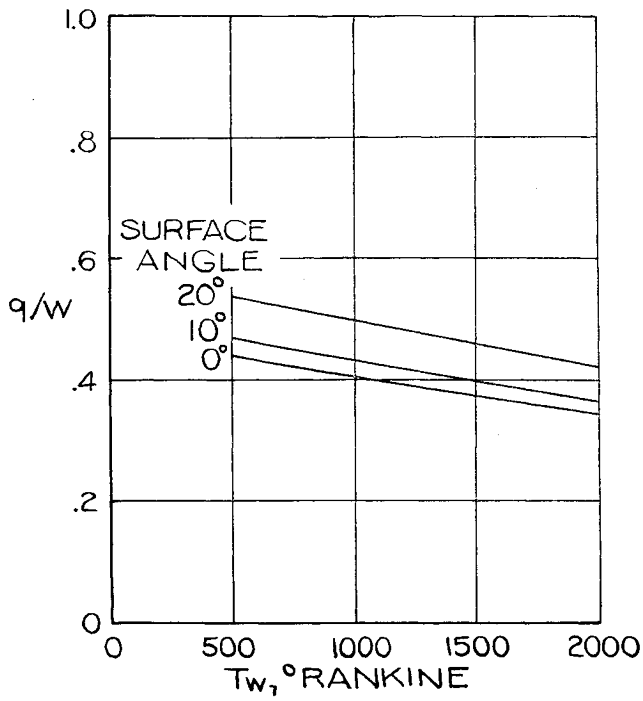

Fig. 1

the skin, causing the skin temperature to vary from a low value at the coolant input to the material-limit temperature at the coolant exit. The recovered energy is, therefore, represented by the average along a coolant tube and is greater than that indicated by the structural material-temperature limit.

A Method of Structural Weight Minimization Suitable for High-Speed Digital Computers

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Let the input for a structural design problem, i.e., lengths, thicknesses, densities, etc., consist of the variables $x_1,x_2, \ldots x_m$ or, in vector form, $x = (x_1,x_2, \ldots x_m)$ and the output be $\sigma = (\sigma_1,\sigma_2, \ldots \sigma_t)$, where each $\sigma_i = \sigma_i(x_1,x_2, \ldots x_m)$ is a function of the x's. The first n_1 of the σ_i may be stresses or deflections, say, at some point of the structure for load condition 1, the next n_1 for condition 2, etc., so that all applied load conditions are represented in the output vector.

The output may have bounds $\sigma^B = (\sigma_1{}^B,\sigma_2{}^B, \ldots \sigma_t{}^B)$, these being in some cases upper and in others lower bounds. If some output σ_i has both, it may appear twice in the output vector. Also define a weight function $w = w(x_1,x_2, \ldots x_m)$, a unit "direction" vector $u = \mathrm{col}(u_1,u_2, \ldots u_m)$, and a gap vector whose ith component is

$$\Delta\sigma_i = \epsilon_1(\sigma_i{}^B - \sigma_i) \qquad (1)$$

where $\epsilon_1 = +1$ if $\sigma_i{}^B$ is an upper bound, and $\epsilon_1 = -1$ if $\sigma_i{}^B$ is a lower bound. The gap vector is so chosen that if a component is negative a limiting design condition has been exceeded.

To start things off, some point x must be selected. It is immaterial at what point this start is made. Then in the neighborhood of this point an approximation is made of the partial derivative matrix

$$R = (r_{ij}) \qquad r_{ij} = \partial\sigma_j/\partial x_i \qquad \begin{matrix} i = 1,2, \ldots , m \\ j = 1,2, \ldots , t \end{matrix} \qquad (2)$$

The calculation of this matrix can be facilitated greatly by use of the "equivalent load" method described later. Also, the gap vector is computed.

It should be noted that, if all the input parameters are altered in the same proportion as the components of some vector u, i.e., if one moves on the input "surface" in the "direction" u, then the change in the vector σ is given by the formula

$$\partial\sigma/\partial u = \mathrm{col}(\partial\sigma_1/\partial u, \partial\sigma_2/\partial u, \ldots \partial\sigma_t/\partial u) = R^T u \qquad (3)$$

The first direction is the normalized gradient of the weight vector

$$u = g^* = g/(g^T)^{1/2}$$
$$g = \mathrm{col}(\partial w/\partial x_1, \partial w/\partial x_2, \ldots , \partial w/\partial x_m) \qquad (4)$$

If none of the gap vector's components are negative, then a reduction in weight is possible, so that the negative of the gradient (steepest descent) is followed until some gap is closed. To estimate the distance in the negative of any direction u that must be moved to close the smallest gap, assume that R will be constant; i.e., that a linear relationship will hold, and use the formula

$$\Delta\lambda = \epsilon_1\epsilon_2 \min_i \left\{ \frac{\Delta\sigma_i}{|\partial\sigma_i/\partial u|} \right\} \qquad (5)$$

where ϵ_2 is the sign of $\partial\sigma_i/\partial u$, and $\Delta\lambda$ is the step length. Since the gaps, in general, will not close linearly with motion along u, a method of successive approximations must be used to close each gap. Actually, more than one gap might close. Suppose r gaps $i_1,i_2, \ldots i_r$ are closed. One then recomputes the R matrix and the gap vector for the new point $x = (x_1 - \Delta\lambda u_1,x_2 - \Delta\lambda u_2, \ldots x_m - \Delta\lambda u_m)$ and also forms a matrix C from the closed-gap columns $i_1,i_2, \ldots i_r$ of the R matrix. Now consider the vector space SPC formed from these columns as basis and also the space SPO orthogonal to SPC. If input is modified in any direction lying in this orthogonal vector space, the gaps will not be altered, at least until linearity is lost. In the present case it is desirable to choose that direction lying in SPO which will reduce the weight as fast as possible. This direction turns out to be the projection of the weight gradient upon the vector space SPO. It is given by the formula

$$u = v(v^Tv)^{1/2} \qquad v = g - C(C^TC)^{-1}C^Tg \qquad (6)$$

The next step is to move along the negative of this projected gradient until the next smallest gap or gaps are closed and then repeat the procedure until as many gaps are closed as possible. This gives minimum weight.

The equivalent load method mentioned earlier is based on the following idea. Let the stiffness matrix equation of the structure be

$$Sw = p \qquad (7)$$

where

w = deflection vector
p = load vector
S = stiffness matrix

Then if a small change ΔS is made in S, this equation becomes

$$(S + \Delta S)\bar{w} = p \qquad (8)$$

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where one now writes $\bar{w}$ for the new deflection vector. Eq. (8) can be written as

$$S\bar{w} = p - \Delta S\bar{w} \tag{9}$$

or

$$\bar{w} = F(p - \Delta S\bar{w}) \tag{10}$$

where $F = S^{-1}$.

To solve this equation for $\bar{w}$, an iteration procedure can be used:

$$w^{(\nu+1)} = F(p - \Delta S \cdot w^{(\nu)}) \tag{11}$$

with $w^{(0)} = w$ in Eq. (7). When ΔS is small, as when calculating an approximation of a partial derivative, the convergence is rapid. Note that the structural change term $-\Delta S w^{(\nu)}$ is treated as a load, which removes the necessity of recomputing the inverse of the altered structural stiffness matrix.

Thermoelastic Differential Equations for Shells of Arbitrary Shape

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Donnell's equations for the analysis of cylindrical shells have been extended to include the effects of arbitrary temperature distributions in Ref. 1. A special case of these equations are those for flat plates which are given in Ref. 2. Although there is no difficulty in further extending the treatment to shells of arbitrary shape, it is desirable to record for future reference the necessary formulation of the Vlasov-type equations for these problems, in view of the considerable interest in thermoelastic problems of shells.

Formulation of Thermoelastic Equations for General Shells

With the notation of Ref. 3, the five equations of equilibrium of a thin shell of arbitrary shape are given by

$$\frac{\partial(A_2T_1)}{\partial\alpha_1} + \frac{\partial(A_1S)}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}S - \frac{\partial A_2}{\partial\alpha_1}T_2 = 0 \tag{1a}$$

$$\frac{\partial(A_2S)}{\partial\alpha_1} + \frac{\partial(A_1T_2)}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1}S - \frac{\partial A_1}{\partial\alpha_2}T_1 = 0 \tag{1b}$$

$$\frac{1}{A_1A_2}\left[\frac{\partial(A_2N_1)}{\partial\alpha_1} + \frac{\partial(A_1N_2)}{\partial\alpha_2}\right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q = 0 \tag{1c}$$

$$N_1 = \frac{1}{A_1A_2}\left[\frac{\partial(A_2M_1)}{\partial\alpha_1} + \frac{\partial(A_1H)}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}H - \frac{\partial A_2}{\partial\alpha_1}M_2\right] \tag{1d}$$

$$N_2 = \frac{1}{A_1A_2}\left[\frac{\partial(A_2H)}{\partial\alpha_1} + \frac{\partial(A_1M_2)}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1}H - \frac{\partial A_1}{\partial\alpha_2}M_1\right] \tag{1e}$$

where transverse shearing forces N_1 and N_2 and tangential surface forces; q_1 and q_2 have been disregarded in Eqs. (1a) and (1b).

The forces and moments are related to the strains and curvature changes of the middle surface by the following equations:

$$\begin{aligned}
T_1 &= \frac{E\delta}{1-\mu^2}(\epsilon_1 + \mu\epsilon_2) - \frac{N_T}{1-\mu}\\
T_2 &= \frac{E\delta}{1-\mu^2}(\epsilon_2 + \mu\epsilon_1) - \frac{N_T}{1-\mu}\\
S &= \frac{E\delta}{2(1+\mu)}\omega\\
M_1 &= \frac{E\delta^3}{12(1-\mu^2)}(k_1 + \mu k_2) - \frac{M_T}{1-\mu}\\
M_2 &= \frac{E\delta^3}{12(1-\mu^2)}(k_1 + \mu k_2) - \frac{M_T}{1-\mu}\\
H &= \frac{E\delta^3}{12(1+\mu)}\tau
\end{aligned} \tag{2}$$

where

$$N_T = \alpha E\int_{-\delta/2}^{\delta/2} T dz \qquad M_T = \alpha E\int_{-\delta/2}^{\delta/2} zT dz \tag{3}$$

where α is the coefficient of linear thermal expansion and T is the arbitrary temperature distribution at the point of interest.

When tangential displacement components in the equations for the changes of curvature and twist are neglected, the curvature change-displacement relations are given by

$$\begin{aligned}
k_1 &= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\\
k_2 &= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\\
\tau &= -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2}\right)
\end{aligned} \tag{4}$$

The substitution of Eqs. (2) and (3) into Eqs. (1d) and (1e) and the neglect of some small terms (see Ref. 3, p. 86) yields

$$\begin{aligned}
N_1 &\cong -\frac{E\delta^3}{12(1-\mu^2)}\frac{1}{A_1}\frac{\partial\Delta w}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{M_T}{1-\mu}\right)\\
N_2 &\cong -\frac{E\delta^3}{12(1-\mu^2)}\frac{1}{A_2}\frac{\partial\Delta w}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{M_T}{1-\mu}\right)\\
\Delta &= \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right)\right]
\end{aligned} \tag{5}$$

By introducing a stress function Φ such that the membrane forces are given by

$$\begin{aligned}
T_1 &= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\Phi}{\partial\alpha_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\frac{\partial\Phi}{\partial\alpha_1}\\
T_2 &= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\Phi}{\partial\alpha_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_2}\frac{\partial\Phi}{\partial\alpha_2}\\
S &= \frac{1}{A_1A_2}\left(\frac{\partial^2\Phi}{\partial\alpha_1\,\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\Phi}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\Phi}{\partial\alpha_2}\right)
\end{aligned} \tag{6}$$

The first two equations of equilibrium (1a) and (1b) can be satisfied approximately when the first-order derivatives of Φ are neglected in comparison with higher order derivatives. Now, the substitution of Eqs. (5) and (6) into Eq. (1c) leads

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